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NOTE

The upper energy cutoff in one-body distributions

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Abstract : It is emphasized that the exact, most probable, one-body distribution function has an inherent upper energy cutoff so that the phase space extends over a finite (rather than infinite) domain Hence, self-bound systems can acquire nice properties as regards the tail region in energy, total mass, stability against evaporation, and maintenance of a temperature gradient. These are compared against the predictions of the standard Maxwellian.

Keywords : Self-bound systems, one-body distributions, upper energy cutoff.

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As is well known, several useful properties of a bound, classical, multiparticle system in equilibrium can be obtained from the one-body probability function f which, in turn, is constructed via the method of most probable distribution (MPD) [1]. The exact, rigorous solutions f of MPD for 2-dimensional, ideal gases placed in a box were recently investigated by Menon and Agrawal [2], and many surprising differences from the conventional Maxwellian solution f_M were noticed. In particular, it was found that the tail of f got abruptly *truncated* at an energy E_K (say) while the Maxwellian tail extended to $+\infty$, but physical applications of this observation were not given in Ref. [2].

The aim of the present paper is three-fold : (i) to generalize the said finding on cutoff to systems interacting through mean fields in arbitrary dimensions, (ii) to describe specific applications of the formalism, and (iii) to pin-point the superiority of our solution f over Maxwell's f_M with regard to each application considered.

Imagine the multiparticle system to be moving in D dimensions under the influence of a mean field W of external or internal origin. Let the symbols

$$m,\ r, p,\ W(r),\ E = p^2/2m + W(r), f(E) \tag{1}$$

respectively, denote the mass, distance, momentum-magnitude, potential energy, full energy, and distribution function of a typical particle. Dividing the energy spectrum into K cells, keeping the total number N of particles as well as the total energy fixed, and maximizing the combinatorial entropy [1], one arrives at the exact variational conditions [2] for classical statistics

$$\psi(n(E)+1) = \ln g(E) + \alpha - E/kT_{th}\,;\; E_1 \le E \le E_K \tag{2a}$$

subject to $$\int^{E_K} dE\, n(E)/\Delta(E) = N. \tag{2b}$$

Here, E_1 to E_K is the single-particle energy spectrum, the cell occupancy n, degeneracy g and level separation Δ are all functions of energy, $\psi(n+1) = d \ln \Gamma(n+1)/dn$ is the digamma function, T_{th} represents the thermodynamic temperature, and the Lagrange parameter α is determined implicitly from the constraint (2b). In the cells of large occupancy $n >> 1$, one usually makes the Stirling approximation $\psi(n+1) \approx \ln n$ to arrive at the standard Maxwellian

$$f_M \equiv n_M/Ng \propto \exp\{-E/kT_{th}\}; \quad E_1 \le E \le +\infty. \tag{3}$$

It was first pointed out by Menon and Agrawal [2] that Stirling approximation must be violated in the cells of small occupation numbers, and we pass on to specific applications of this important fact assuming $T_{th} > 0$ always.

(A) Tail region :

We shall first prove that 'the tail of the actual MPD solution can never be extended to $E = +\infty$'. Indeed, as the energy E goes on increasing in eq. (2a), the term $\ln g(E)$ can grow, at the most, only logarithmically while the term $-E/kT_{th}$ diminishes linearly so that there must exist a *finite* point E_K (say) at which $n(E_K) = 0$, *i.e.*, at which

$$\ln g(E_K) + \alpha - E_K/kT_{th} = \psi(1) = -0.577 \tag{4}$$

The existence of a sharply truncated MPD solution will play a crucial role in the subsequent discussion. Note that the Maxwellian solution (3) becomes *invalid* in the tail region.

(B) Density and mass :

We shall next prove that 'the cutoff E_K is intimately linked with the radius, *i.e.*, size R of the system'. Indeed, in MPD there will exist a turning point R beyond which no classical particle can go implying that the mass density $\rho(r)$ vanishes identically in that domain, *i.e.*,

$$W(R) = E_K; \quad \rho(r) \equiv 0 \quad \text{for} \quad r > R. \tag{5}$$

The total mass M, which is the volume-integral of $\rho(r)$, is clearly finite. Also, if $W(r)$ is short-range attractive, *i.e.*, $W(\infty) = -0$ the MPD system will not evaporate provided $E_K < 0$. In other words, our exact solution f does describe a very large number of self-bound, natural systems like nuclei, atoms, solids and galaxies which have sharp sizes, finite masses, and manifest stability.

Note that the Maxwellian density for short-range mean fields behaves like a constant at large distances :

$$\rho_M(r) \propto \exp\{-W(r)/kT_{\text{th}}\}_{\;r\to\infty} \to \text{constant.} \tag{6}$$

Hence, the Maxwellian system (in absence of an enclosing box) has *infinite* radius, acquires unbounded mass, and goes on ejecting particles into the continuum [3].

(C) Gravitational system :

Next, we shall prove that 'the MPD solution easily handles the $1/r$ tailed potential but the Maxwellian runs into serious trouble even after restricting to the subspace of bound trajectories'. Indeed, our result (5) continues to hold for stellar and galactic systems described by $W(r) \sim 1/r$ at large r. However, if one restricts oneself to the bound spectrum [4] $E \le 0$, the Maxwellian density (labelled by a prime) would behave as

$$\rho'_M(M) \propto \int_0^{p'(r)} dp\, p^{D-1} f_M \underset{r\to\infty}{\sim} r^{-D/2} \tag{7}$$

where $p'(r) \equiv \{-2m\, W(r)\}^{1/2}$. Clearly, the primed mass $\int_0^\infty dr\, r^{D-1}\, \rho'(r)$ again diverges. In other words, Maxwellian galaxies would acquire *infinite* mass and eventually evaporate into nonexistence unless enclosed by a box —a fact which has been a serious difficulty of gravitational statistical mechanics [5].

(D) Temperature profile :

Finally, we shall prove that 'the MPD solution can support a static gradient of the kinetic temperature'. Indeed, the momentum magnitude range now is

$$0 \le p \le p_K(r); \qquad p_K(r) \equiv \{2m\,(E_K - W(r))\}^{1/2} \tag{8}$$

Hence, the kinetic temperature predicted by MPD *viz.*,

$$T_{\text{kin}}(r) \propto \int^{p_K(r)} dp\, p^{D+1} f \Big/ \int^{p_K(r)} p^{D-1} f \tag{9}$$

will, in general, be r-dependent but a heat flow will not occur because f is a function of E only. This, of course, is the case with the tabulated results [6] on the reference terrestrial atmosphere fitted well by Rayleigh's hydrostatic adiabatic model [7] having an inherent energy cutoff [8].

Note that the Maxwellian has a momentum range $0 \leq p \leq \infty$ so that the predicted temperature becomes uniform (*i.e.*, $T^{M}_{\rm kin}(r) = T_{\rm th}$) which *contradicts* the observations in the troposphere.

Before ending, it may be added that all the conclusions of this paper may be generalized to quantum statistics as well.

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